Journal of Statistical Physics, Vol. 24, No. 1, 1981

An Electrothermal Instability in a Conducting Wire: Homogeneous and Inhomogeneous Stationary States for an Exactly Solvable Model

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Received January 8, 1980

An exactly solvable model of the ballast resistor is considered. Analytic expressions are obtained for the nonuniform stationary temperature distributions and the corresponding $I-V$ characteristics. A bifurcation point for Neumann boundary conditions is found and its analytic properties are discussed. It is found that the infinite wire limit plays a role analogous to the thermodynamic limit in statistical mechanics for equilibrium phase transitions.

KEY WORDS: Electrothermal instability; nonuniform stationary states; exact solutions; bifurcation point.

1. INTRODUCTION

In a previous paper[(1)] we discussed an electrothermal instability in a conducting wire, the so-called ballast resistor. The resistivity of the wire, which is surrounded by a gas, is an increasing function of the temperature. Both the temperature of the gas as well as the electric current I through the wire (or the voltage difference V between the end points) are externally controlled. Under suitable conditions an instability occurs and the wire can sustain inhomogeneous stationary temperature distributions. The $I-V$ characteristic corresponding to these inhomogeneous states contains a segment where the current is constant while the voltage varies.

In this paper we shall consider a special solvable model of the ballast

Dedicated to the memory of our colleague and friend Pierre Résibois.

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0022-4715/81/0100-0215$03.00/0

resistor in which the resistivity is zero[2] below a critical temperature T_c and equal to a constant R_0 above T_c. This system corresponds to the hot-spot model for superconducting microbridges introduced by Skocpol *et al.*[(2)] These authors constructed an explicit solution for the case that the temperature at the end points is essentially equal to the temperature of the bath and discuss the relevance of the model in connection with experimental data. Here we study this system for a different class of boundary conditions for which the temperature gradient is zero at the end points of the wire (Neumann boundary conditions). The resulting $I-V$ characteristics have a much richer structure than in the first case (Dirichlet boundary conditions), containing, e.g., a bifurcation point.

On the basis of the analysis in this paper we shall discuss in a forthcoming article the stability properties of the inhomogeneous states.

In Section 2 we discuss some general properties of the model. In Section 3 the stationary temperature distributions are given for Neumann boundary conditions. An infinite set of periodic inhomogeneous solutions is found to exist. For each solution an analytic expression is found for the voltage difference V as a function of the current I. All these solutions converge in the $I-V$ characteristic to a homogeneous solution in the same bifurcation point. We also find that the solutions satisfy a symmetry relation, for which a derivation in a more general case is given in the Appendix. An analysis along the same lines is given in Section 4 for Dirichlet boundary conditions. Finally, the behavior for a long wire is discussed in the last section. It is found that in the limit $L \to \infty$, where L is the length of the wire, V/L becomes a nonanalytic function of I for the inhomogeneous solutions. Thus this limit plays an analogous role to the thermodynamic limit in statistical mechanics for equilibrium phase transitions.

2. THE MODEL; GENERAL PROPERTIES

We consider a thin wire of length L along which an electric current I may flow. The wire is surrounded by a heat bath which is kept at the constant temperature T_B. The temperature distribution as a function of the position and time $T(x, t)$ satisfies the equation

$$c \frac{\partial}{\partial t} T(x,t) = \frac{\partial}{\partial x} \lambda \frac{\partial}{\partial x} T(x,t) - q\left[T(x,t) - T_B\right] + I^2 R \quad \text{for} \quad 0 < x < L \tag{2.1}$$

[2] If the resistivity is a finite constant smaller than R_0 rather than zero, the model is still exactly solvable; see Appendix.

Here c is the specific heat per unit of length, λ is the heat conductivity of the wire, q is the heat transfer coefficient to the heat bath, and R is the resistance of the wire per unit of length. In principle all these quantities depend on the local temperature $T(x,t)$.

In this paper we shall restrict ourselves to the discussion of the stationary states for which the temperature distribution satisfies

$$\frac{d}{dx}\lambda\frac{d}{dx}T(x) - q\left[T(x) - T_B\right] + I^2R = 0 \tag{2.2}$$

In particular we shall consider the following idealized model:

$$R(T) = R_0\Theta(T - T_c) \tag{2.3a}$$

$$q(T) = q_0 \tag{2.3b}$$

$$\lambda(T) = \lambda_0 \tag{2.3c}$$

Here Θ is the Heaviside function, $\Theta(s) = 1$ for $s > 0$ and $\Theta(s) = 0$ for $s < 0$. Equation (2.3a) implies that the wire is superconducting below T_c and has a constant resistivity above T_c. The coefficients q and λ are assumed to be constants.[3]

Without loss of generality we may take the zero of the temperature scale at the bath temperature, or equivalently we put

$$T_B = 0 \tag{2.4}$$

in Eqs. (2.1) and (2.2). Furthermore, we introduce the following dimensionless parameter instead of the position x:

$$y \equiv (q/\lambda)^{1/2}x \qquad \text{with} \quad 0 \leqslant y \leqslant (q/\lambda)^{1/2}L \equiv y_L \tag{2.5}$$

We also define the following temperature:

$$T_h \equiv I^2R_0/q \tag{2.6}$$

Using Eqs. (2.3)–(2.6), one finds from Eq. (2.2) the following equation for the stationary states:

$$\frac{d^2}{dy^2}T(y) - T(y) + T_h\Theta(T(y) - T_c) = \frac{d^2}{dy^2}T(y) + \frac{d}{dT}\phi(T) = 0 \tag{2.7}$$

where the "potential" ϕ is defined by

$$\phi(T) \equiv \int_0^T \left[T_h\Theta(\tau - T_c) - \tau\right]d\tau \tag{2.8}$$

It follows from Eq. (2.7) that the following function is a constant along the

[3] It is easy to extend the subsequent analysis to the case that q and λ have different values below and above T_c.

wire[1] in the stationary state:

$$H \equiv \tfrac{1}{2}(dT/dy)^2 + \phi(T) \tag{2.9}$$

The potential may easily be calculated and one finds

$$\phi(T) = T_h(T - T_c)\Theta(T - T_c) - \tfrac{1}{2}T^2 = \begin{cases} -\tfrac{1}{2}T^2 & \text{for } T < T_c \\ \phi_h - \tfrac{1}{2}(T - T_h)^2 & \text{for } T > T_c \end{cases} \tag{2.10}$$

where

$$\phi_h \equiv \tfrac{1}{2}T_h(T_h - 2T_c) \tag{2.11}$$

The potential is parabolic for $T > T_c$ as well as for $T < T_c$. The point $T = T_c$ is a common point of both parabolas. We also see that the potential has one maximum if $T_c > T_h = I^2R_0/q$. If $0 < T_c < T_h$ the potential has two maxima, at $T = 0$ and at T_h, and one minimum, $\phi_c \equiv -\frac{1}{2}T_c^2$ at T_c. See also Fig. 1. The possibility of having two maxima for sufficiently large values of the current I plays an essential role in our analysis of the nature of the stationary states.

In order to make the description of the model complete, boundary conditions at $x = 0$ and $x = L$ (or alternatively at $y = 0$ and $y = y_L$) must be specified. In the following sections we shall consider Neumann boundary conditions.

$$\left.\frac{dT}{dy}\right|_{y=0} = \left.\frac{dT}{dy}\right|_{y=y_L} = 0 \tag{2.12}$$

as well as Dirichlet boundary conditions

$$T(0) = T(y_L) = T_B = 0 \tag{2.13}$$

The latter are usually satisfied in good approximation in ballast resistor experiments.

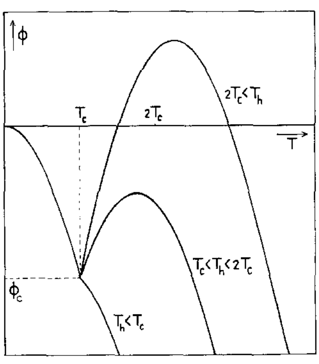

Fig. 1. The potential function $\phi(T)$ for three values of $T_h = R_0I^2/q$.

Along the wire one may distinguish two different regions:

(i) *Cold sections, $T(y) < T_c$*. In these sections Eq. (2.7) reduces to

$$\frac{d^2}{dy^2} T = T \tag{2.14}$$

with solutions of the general form

$$T(y) = A_1 e^y + A_2 e^{-y} \tag{2.15}$$

(ii) *Hot sections*, $T(y) > T_c$. In these sections Eq. (2.7) reduces to

$$\frac{d^2}{dy^2} T = T - T_h \tag{2.16}$$

with solutions of the general form

$$T(y) = T_h + A_3 e^y + A_4 e^{-y} \tag{2.17}$$

The amplitudes A_1, A_2, A_3, and A_4 must be chosen such that at the end points of the wire the boundary conditions are satisfied. Furthermore, the cold and the hot sections should be joined together in such a way that Eq. (2.7) is satisfied also at the transition points y_c. This is the case if

$$T \quad \text{and} \quad dT/dy \quad \text{are continuous at} \quad y = y_c \tag{2.18}$$

Using the conditions specifying the cold and the hot regions and the continuity of T, it follows immediately that

$$T(y_c) = T_c \tag{2.19}$$

Furthermore, we note that if the amplitudes in Eqs. (2.15) and (2.17) are chosen such that the corresponding temperature distributions give rise to the same value of H [cf. Eq. (2.9)], then dT/dy is automatically continuous in view of the continuity of T in y_c.

In our analysis we also need the expression for the voltage difference V between the end points of the wire,

$$V = \int_0^L R(T(x)) I \, dx = (\lambda/q)^{1/2} I R_0 \int_0^{y_L} \Theta(T(y)) \, dy \tag{2.20}$$

Obviously V is proportional to the total length of the hot ($T > T_c$) sections along the wire.

3. STATIONARY STATES FOR NEUMANN BOUNDARY CONDITIONS

We first give the homogeneous solutions of the equation for the stationary states, Eq. (2.7). As is obvious from this equation, homogeneous solutions exist for temperatures such that the potential ϕ has a maximum.

This leads to one homogeneous solution

$$T(y) = 0 \qquad \text{with} \quad V = 0 \tag{3.1}$$

for all values of T_h and therefore of the current. If $T_h > T_c$ there exists an additional homogeneous solution

$$T(y) = T_h = I^2 R_0/q \qquad \text{with} \quad V = IR_0 L \tag{3.2}$$

Note that both stationary homogeneous solutions satisfy the Neumann boundary conditions.

We now consider inhomogeneous stationary temperature distributions. We shall first study distributions with one cold section for $0 < y < y_c$ and one hot section for $y_c < y < y_L$. The distributions in these sections are given by

$$T(y) = (-2H)^{1/2} \cosh y \qquad \text{for} \quad 0 \leqslant y \leqslant y_c \tag{3.3}$$

$$T(y) = T_h - \left[2(\phi_h - H)\right]^{1/2} \cosh(y_L - y_c) \qquad \text{for} \quad y_c \leqslant y \leqslant y_L \tag{3.4}$$

These solutions are indeed of the form given in Eqs. (2.15) and (2.16), respectively. Moreover, they satisfy the Neumann boundary conditions, Eq. (2.12). The amplitudes, which depend on the "conserved" quantity H, have been chosen such that Eq. (2.9) is satisfied in both sections. An immediate consequence of Eqs. (3.3) and (3.4) is that these solutions only exist if

$$H < 0 \qquad \text{and} \qquad H < \phi_h \tag{3.5}$$

At the transition point y_c between the cold and the hot sections the temperature is equal to T_c [cf. Eq. (2.19)], and hence

$$(-2H)^{1/2} \cosh y_c = T_c = T_h - \left[2(\phi_h - H)\right]^{1/2} \cosh(y_L - y_c) \tag{3.6}$$

It follows from the second equality that

$$T_h > T_c \tag{3.7}$$

which is the condition for the existence of two maxima in the potential. The inhomogeneous solution therefore only exists for currents such that the potential has two maxima.

Solving y_c and $y_L - y_c$ from Eq. (3.6) one obtains

$$y_c = \operatorname{arc\,cosh}\left[(\phi_c/H)^{1/2}\right] \tag{3.8}$$

$$y_L - y_c = \operatorname{arc\,cosh}\left\{\left[(\phi_h - \phi_c)/(\phi_h - H)\right]^{1/2}\right\} \tag{3.9}$$

where

$$\phi_c \equiv \phi(T_c) = -\tfrac{1}{2} T_c^2 \tag{3.10}$$

It follows from these equations that

$$H > \phi_c \tag{3.11}$$

Adding Eqs. (3.8) and (3.9), one obtains

$$\begin{aligned} y_L &= \operatorname{arc\,cosh}\left[(\phi_c/H)^{1/2}\right] + \operatorname{arc\,cosh}\left\{\left[(\phi_h - \phi_c)/(\phi_h - H)\right]^{1/2}\right\} \\ &= \operatorname{arc\,cosh}\left\{\left[-H(\phi_h - H)\right]^{-1/2}\left(H + \tfrac{1}{2}T_c T_h\right)\right\} \end{aligned} \tag{3.12}$$

where we used the addition theorem

$$|\operatorname{arc\,cosh} x \pm \operatorname{arc\,cosh} y| = \operatorname{arc\,cosh}\left[xy \pm (x^2 - 1)^{1/2}(y^2 - 1)^{1/2}\right] \tag{3.13}$$

Equation (3.12) for H may be solved by inversion and one obtains one solution

$$\begin{aligned} H(I; L) &= H(z; y_L) \\ &= -T_c^2\left[1 + z\sinh^2 y_L + (\cosh y_L)\left(1 + z^2\sinh^2 y_L\right)^{-1/2}\right]^{-1} \end{aligned} \tag{3.14}$$

where we have introduced the variable

$$z(I) \equiv 1 - \frac{2T_c}{T_h} = 1 - \frac{2qT_c}{R_0 I^2} \Rightarrow I = \left(\frac{2qT_c/R_0}{1 - z}\right)^{1/2} \tag{3.15}$$

The solution has been chosen consistent with the inequalities given in (3.5) and (3.11). In terms of the new variable one has

$$\phi_h = 2T_c^2 z(1 - z)^{-2} \tag{3.16}$$

Note that for the inhomogeneous solution, $T_h > T_c$; consequently, $-1 < z < 1$, while furthermore $\operatorname{sign}\phi_h = \operatorname{sign} z$.

Substituting the solution (3.14) into Eqs. (3.8) and (3.9), one obtains for the lengths of the cold and the hot sections

$$y_c(z; y_L) = \tfrac{1}{2}\operatorname{arc\,cosh}\left[z\sinh^2 y_L + (\cosh y_L)\left(1 + z^2\sinh^2 y_L\right)^{1/2}\right] \tag{3.17a}$$

$$y_L - y_c(z; y_L) = \tfrac{1}{2}\operatorname{arc\,cosh}\left[-z\sinh^2 y_L + (\cosh y_L)\left(1 + z^2\sinh^2 y_L\right)^{1/2}\right] \tag{3.17b}$$

where we have used Eq. (3.13) for $x = y$. It follows from these equations that

$$y_c(z; y_L) = y_L - y_c(-z; y_L) \tag{3.18}$$

This implies that the transformation $z \to -z$, i.e., $I \to I' = I[(I^2 R_0/qT_c) - 1]^{-1/2}$, interchanges the length of the hot and the cold sections. Note that for $z = 0$ the length of the hot and the cold sections are both equal to $\frac{1}{2}y_L$.

This relation suggests the introduction of the following parameter:

$$\gamma \equiv 2y_c/y_L - 1 \qquad \text{with} \quad -1 \leqslant \gamma \leqslant 1 \tag{3.19}$$

This implies that

$$y_c = \tfrac{1}{2}(1+\gamma)y_L \qquad \text{and} \qquad y_L - y_c = \tfrac{1}{2}(1-\gamma)y_L \tag{3.20}$$

It follows from Eq. (3.17) that $\operatorname{sign}\gamma = \operatorname{sign} z$. Using Eqs. (3.13) and (3.17a) and the identity $|\operatorname{arc\,sinh} x| = \operatorname{arc\,cosh}[(x^2+1)^{1/2}]$, one then obtains

$$\gamma(z; y_L) = (1/y_L)\operatorname{arc\,sinh}(z \sinh y_L) \tag{3.21}$$

This parameter is an antisymmetric function of z,

$$\gamma(z; y_L) = -\gamma(-z; y_L) \tag{3.22}$$

which is equivalent to Eq. (3.18).

The voltage difference between the end points of the wire for the above solution is given by [cf. Eq. (2.20)]

$$\begin{aligned} V(I;L) &= IR_0(\lambda/q)^{1/2}[y_L - y_c(z;y_L)] = \tfrac{1}{2}IR_0L[1-\gamma(z;y_L)] \\ &= \tfrac{1}{2}IR_0L[1-(1/y_L)\operatorname{arc\,sinh}(z\sinh y_L)] \end{aligned} \tag{3.23}$$

One may immediately evaluate the value of V in the following three cases, corresponding to $z = -1$, $z = 0$, and $z \to 1$:

$$V\big(I = (qT_c/R_0)^{1/2}; L\big) = L(qT_cR_0)^{1/2} = LR_0I \tag{3.24a}$$

$$V\big(I = (2qT_c/R_0)^{1/2}; L\big) = L(\tfrac{1}{2}qT_cR_0)^{1/2} = \tfrac{1}{2}LR_0I \tag{3.24b}$$

$$\lim_{I\to\infty} V(I;L) = 0 \tag{3.24c}$$

It follows from Eq. (3.23) and the antisymmetry of γ [Eq. (3.22)] that

$$V(I;L)/I + V(I';L)/I' = R_0L \tag{3.25}$$

with

$$I' = I\big[(I^2R_0/qT_c) - 1\big]^{-1/2} \tag{3.26}$$

The derivative of V with respect to I is given by

$$\begin{aligned} \frac{dV}{dI} &= \frac{V}{I} - \frac{1}{2}IR_0L\frac{d}{dI}\gamma(z;y_L) \\ &= \frac{V}{I} - R_0\left(\frac{\lambda}{q}\right)^{1/2}(\sinh y_L)\left[(1-z)(1+z^2\sinh^2 y_L)^{-1/2}\right] \end{aligned} \tag{3.27}$$

One may again evaluate dV/dI in the three cases corresponding to $z = -1$,

$z = 0$, and $z = 1$:

$$\left.\frac{dV}{dI}\right|_{I=(qT_c/R_0)^{1/2}} = R_0 L\left(1 - \frac{2}{y_L}\tanh y_L\right) \tag{3.28a}$$

$$\left.\frac{dV}{dI}\right|_{I=(2qT_c/R_0)^{1/2}} = R_0 L\left(1 - \frac{1}{y_L}\sinh y_L\right) \tag{3.28b}$$

$$\lim_{I\to\infty} \frac{dV}{dI} = 0 \tag{3.28c}$$

It follows furthermore from Eqs. (3.27) and (3.23) that

$$dV/dI < V/I < R_0 L \tag{3.29}$$

One may also show a stronger inequality for $z > 0$:

$$dV/dI \leqslant 0 \qquad \text{for} \quad I \geqslant (2T_c q/R_0)^{1/2} \tag{3.30}$$

This inequality follows by proving the fact that $(d/dz)(dV/dI) > 0$ for $0 \leqslant z \leqslant 1$ and by observing that dV/dI approaches zero [cf. Eq. (3.28c)] for $z \to 1$.

It is interesting to consider in more detail the behavior of the solution for $z\downarrow -1$, i.e., $T_h \downarrow T_c$ or $I\downarrow(qT_c/R_0)^{1/2}$. In this limit the voltage [cf. Eq. (3.20a)] approaches the value of the voltage for the homogeneous solution [cf. Eq. (3.2)] for the same value of the current. Consequently, the point

$$I = (qT_c/R_0)^{1/2}, \qquad V = L(qT_c R_0)^{1/2} \tag{3.31}$$

is a bifurcation point in the $I-V$ characteristic. As follows from Eqs. (3.28a) and (3.7), the two branches for the inhomogeneous and the homogeneous solutions approach the bifurcation point from a different direction. One may show that the temperature distributions of both the inhomogeneous as well as the homogeneous solution become identical at the bifurcation point

$$\lim_{I\downarrow(qT_c/R_0)^{1/2}} T(y) = T_c \tag{3.32}$$

Finally we consider inhomogeneous solutions with more than one cold or hot section. Solutions of this nature may easily be constructed by considering an inhomogeneous solution of the type given above on a wire of length L/n, where n is an integer. By repetition of this solution on the n segments of the wire of length L, one obtains a solution with several hot and cold sections which satisfies the boundary conditions. The solutions are periodic with a period $2L/n$. One may easily show that such periodic solutions are the only inhomogeneous solutions that exist (cf. also Ref. 1). Most of the properties of the periodic solutions for $n \geqslant 2$ are directly related to those for $n = 1$. In particular, the value H_n for these solutions is given by

$$H_n(I;L) = H(z; y_L/n) \tag{3.33}$$

in terms of the solution given in Eq. (3.14) for the $n = 1$ case.

The voltage difference between the end points of the wire for the periodic solution is given by [cf. Eq. (2.20)]

$$\begin{aligned} V_n(I;L) &= IR_0(\lambda/q)^{1/2}\left[y_L - ny_c(z; y_L/n)\right] \\ &= \tfrac{1}{2}IR_0L\left[1 - \gamma(z; y_L/n)\right] \\ &= \tfrac{1}{2}IR_0L\left\{1 - \frac{n}{y_L}\operatorname{arc\,sinh}\left[z\sinh(y_L/n)\right]\right\} \end{aligned} \tag{3.34}$$

The derivative of V_n with respect to I becomes

$$\frac{dV_n}{dI} = \frac{V_n}{I} - nR_0\left(\frac{\lambda}{q}\right)^{1/2}(1-z)\left(\sinh\frac{y_L}{n}\right)\left(1 + z^2\sinh^2\frac{y_L}{n}\right)^{-1/2} \tag{3.35}$$

It follows as in the $n = 1$ case that all periodic solutions converge to the homogeneous solution at the same bifurcation point given by $I = (qT_c/R_0)^{1/2}$, $V = L(qT_cR_0)^{1/2}$ in the I–V characteristic. Instead of Eq. (3.28a) one now obtains

$$\left.\frac{dV_n}{dI}\right|_{I=(qT_c/R_0)^{1/2}} = R_0L\left(1 - \frac{2n}{y_L}\tanh\frac{y_L}{n}\right) \tag{3.36}$$

so that all branches approach the bifurcation point from a slightly different direction. In the limit $n \to \infty$ one finds

$$\lim_{n\to\infty}\left.\frac{dV}{dI}\right|_{i=(qT_c/R_0)^{1/2}} = -R_0L \tag{3.37}$$

One may also study the behavior of the solutions for $n \to \infty$ for all values of the current larger than $(qT_c/R_0)^{1/2}$. Using Eqs. (3.3), (3.4), (3.14), and (3.17), with y_L/n instead of y_L, it follows that

$$\lim_{n\to\infty} T(y) = T_c \tag{3.38}$$

The $n \to \infty$ value is therefore again homogeneous and is the analog in the present case of the unstable homogeneous solution discussed in Ref. 1. It should be emphasized, however, that $T(y) = T_c$ is not a real solution of Eq. (2.7). Using Eq. (3.34), one finds

$$\lim_{n\to\infty} V_n(I;L) = IR_0LT_c/T_h = qLT_c/I \tag{3.39}$$

which determines this limiting I–V characteristic.

If one chooses the current such that $z = 0$, i.e., $I = (2T_cq/R_0)^{1/2}$, it follows from Eq. (3.34) that

$$V_n\left(I = (2T_cq/R_0)^{1/2}; L\right) = \tfrac{1}{2}L(2T_cqR_0)^{1/2} = \tfrac{1}{2}LR_0I \tag{3.40}$$

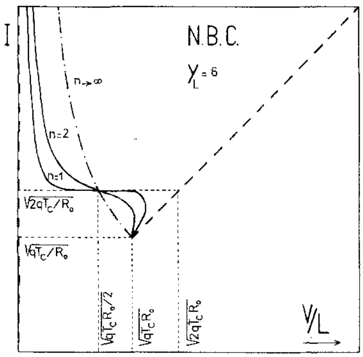

Fig. 2. The $I-V$ characteristics for Neumann boundary conditions. The length of the wire has been chosen such that $y_L = 6$. The dashed lines correspond to the homogeneous solutions. The solid lines correspond to the inhomogeneous solutions for $n = 1$ and $n = 2$. The dash–dotted line corresponds to the $n \to \infty$ limit.

This implies that all characteristics cross each other at the point $I = (2T_cq/R_0)^{1/2}$, $V = \frac{1}{2}L(2T_cqR_0)^{1/2}$. It follows from Eqs. (3.34), (3.35), and (3.40) that the derivatives at this point are given by

$$\left.\frac{dV_n}{dI}\right|_{I=(2T_cq/R_0)^{1/2}} = \frac{1}{2}R_0L\left(1 - 2\frac{n}{y_L}\sinh\frac{y_L}{n}\right) \qquad (3.41)$$

In view of the fact that the function y^{-1} arc sinh($|z|$sinh y) is a monotonically increasing function of y for $0 < |z| < 1$, one has

$$\begin{aligned} V_{n+1}(I) &> V_n(I) \qquad \text{for} \quad 0 < z(I) < 1 \\ V_{n+1}(I) &< V_n(I) \qquad \text{for} \quad -1 < z(I) < 0 \end{aligned} \qquad (3.42)$$

The corresponding $I-V$ characteristics are plotted in Fig. 2 for the $n = 1, 2$ and $n \to \infty$ solutions, choosing a small value of y_L.

4. STATIONARY STATES FOR DIRICHLET BOUNDARY CONDITIONS

Let us now study the solutions of Eq. (2.7) with Dirichlet boundary conditions, Eq. (2.13). In this case there is only one homogeneous solution

$$T(y) = 0 \qquad \text{with} \quad V = 0 \qquad (4.1)$$

In addition, inhomogeneous solutions exist.

We shall now consider solutions with cold sections for $0 < y < y_c$ and $y_L - y_c < y < y_L$ and a hot section for $y_c < y < y_L - y_c$ which are symmetric around the center of the wire. The temperature distribution for these solutions is

$$T(y) = (2H)^{1/2} \sinh y, \qquad \text{for} \quad 0 \leqslant y \leqslant y_c \quad (4.2a)$$

$$= T_h - [2(\phi_h - H)]^{1/2} \cosh(\tfrac{1}{2} y_L - y), \qquad \text{for} \quad y_c \leqslant y \leqslant y_L - y_c \quad (4.2b)$$

$$= (2H)^{1/2} \sinh(y_L - y), \qquad \text{for} \quad y_L - y_c \leqslant y \leqslant y_L \quad (4.2c)$$

These solutions are again of the form given in Eqs. (2.15) and (2.16), respectively, and satisfy Dirichlet boundary conditions, Eq. (2.13). The amplitudes, which depend on the "conserved" quantity H, have been chosen in such a way that Eq. (2.9) is satisfied everywhere in the wire. At the end of this section we shall show that all inhomogeneous solutions for Dirichlet boundary conditions are of the form given in Eq. (4.2). A consequence of Eq. (4.2) is that these solutions only exist if

$$0 < H < \phi_h \tag{4.3}$$

It follows from this inequality that $T_h > 2T_c$. At the transition points y_c and $y_L - y_c$ between the hot and the cold sections the temperature is equal to T_c [cf. Eq. (2.19)], and hence

$$(2H)^{1/2} \sinh y_c = T_c = T_h - [2(\phi_h - H)]^{1/2} \cosh(\tfrac{1}{2} y_L - y_c) \tag{4.4}$$

Solving y_c and $\frac{1}{2}y_L - y_c$ from this equation, one obtains

$$y_c = \operatorname{arc\,sinh}\left[(-\phi_c/H)^{1/2}\right] \tag{4.5}$$

$$\frac{1}{2} y_L - y_c = \operatorname{arc\,cosh}\left[\left(\frac{\phi_h - \phi_c}{\phi_h - H}\right)^{1/2}\right] \tag{4.6}$$

Adding the last two equations, one obtains

$$\frac{1}{2} y_L = \operatorname{arc\,sinh}\left[\left(-\frac{\phi_c}{H}\right)^{1/2}\right] + \operatorname{arc\,cosh}\left[\left(\frac{\phi_h - \phi_c}{\phi_h - H}\right)^{1/2}\right]$$

$$= \operatorname{arc\,sinh}\left\{[H(\phi_h - H)]^{-1/2}(H + \tfrac{1}{2} T_c T_h)\right\} \tag{4.7}$$

where we have used the identity

$$\operatorname{arc\,sinh} x \pm \operatorname{arc\,cosh} y = \operatorname{arc\,sinh}\left[xy \pm (x^2 + 1)^{1/2}(y^2 - 1)^{1/2}\right] \tag{4.8}$$

Equation (4.7) for H may be solved by inversion and one obtains two solutions for each value of the current:

$$H_{\pm} = T_c^2 \Big\{ z\cosh^2(y_L/2) - 1 \pm \sinh(y_L/2)\left[z^2\cosh^2(y_L/2) - 1\right]^{1/2}\Big\}^{-1} \tag{4.9}$$

which are real and positive [cf. Eq. (4.3)] if

$$z(I) = 1 - 2T_c/T_h = 1 - 2qT_c/R_0 I^2 \geqslant \cosh^{-1}(y_L/2) \equiv z_{\min} \tag{4.10}$$

Substituting H into Eq. (4.5), one obtains

$$y_c^{\pm} = \tfrac{1}{2}\operatorname{arc\,cosh}\Big\{ z\cosh^2(y_L/2) \pm \sinh(y_L/2)\left[z^2\cosh^2(y_L/2) - 1\right]^{1/2}\Big\} \tag{4.11}$$

where we used the identity $|\operatorname{arc\,sinh} x| = \frac{1}{2}\operatorname{arc\,cosh}(2x^2 + 1)$. Using Eq. (3.13), one may easily show that

$$y_c^- + y_c^+ = \tfrac{1}{2} y_L \tag{4.12}$$

This formula implies that the total lengths of the cold and the hot sections are interchanged if one goes from the minus to the plus solution. It also follows from Eq. (4.11) that

$$0 \leqslant y_c^- \leqslant \tfrac{1}{4} y_L \leqslant y_c^+ \leqslant \tfrac{1}{2} y_L \tag{4.13}$$

We introduce again a parameter γ,

$$\gamma \equiv 4y_c^+/y_L - 1 = -4y_c^-/y_L + 1 \qquad \text{with} \quad 0 \leqslant \gamma \leqslant 1 \tag{4.14}$$

This implies that

$$y_c^{\pm} = \tfrac{1}{4} y_L (1 \pm \gamma) \tag{4.15}$$

It follows from Eq. (4.11), using also Eq. (3.13), that

$$\gamma = (2/y_L)\operatorname{arc\,cosh}\left[z\cosh(y_L/2)\right] \tag{4.16}$$

The voltage difference between the end points of the wire is given by [cf. Eq. (2.20)]

$$V^{\pm} = IR_0(\lambda/q)^{1/2}(y_L - 2y_c^{\pm}) = \tfrac{1}{2} IR_0 L(1 \mp \gamma) = \tfrac{1}{2} IR_0 L\left\{1 \mp (2/y_L)\operatorname{arc\,cosh}\left[z\cosh(y_L/2)\right]\right\} \tag{4.17}$$

It follows immediately that

$$V^+ + V^- = IR_0 L \tag{4.18}$$

One may evaluate the voltage for $z \to 1$ and one finds

$$\lim_{I\to\infty} \left[V^{\pm}(I;L)/I \right] = \tfrac{1}{2} R_0 L(1 \mp 1) \tag{4.19}$$

The two solutions become identical for the lowest value $z_{\min}$ of z [cf. Eq. (4.10)]. The corresponding value of the voltage is given by

$$\begin{aligned} &V^{\pm}\left(I = (2qT_c/R_0)^{1/2}\left[1 - \cosh^{-1}(y_L/2)\right]^{-1/2}, L\right) \\ &= \tfrac{1}{2} I R_0 L \\ &= \tfrac{1}{2} L (2qT_c R_0)^{1/2}\left[1 - \cosh^{-1}(y_L/2)\right]^{-1/2} \end{aligned} \tag{4.20}$$

The derivative of V with respect to the current becomes

$$\frac{dV^{\pm}}{dI} = \frac{V^{\pm}}{I} \mp 2R_0\left(\frac{\lambda}{q}\right)^{1/2}(1-z)\left(\cosh\frac{y_L}{2}\right)\left(z^2\cosh^2\frac{y_L}{2} - 1\right)^{-1/2} \tag{4.21}$$

For $z \to 1$ one may again evaluate this derivative and one obtains, using also Eq. (4.19),

$$\lim_{I\to\infty} \frac{dV^{\pm}}{dI} = \lim_{I\to\infty} \frac{V^{\pm}}{I} = \frac{1}{2} R_0 L(1 \mp 1) \tag{4.22}$$

At $z = z_{\min}$ one finds

$$\left.\frac{dI}{dV}\right|_{z=z_{\min}} = \left.\left(\frac{dV^{\pm}}{dI}\right)^{-1}\right|_{z=z_{\min}} = 0 \tag{4.23}$$

The current has a minimum as function of the voltage for the voltage given in Eq. (4.20). One may prove the following inequalities:

$$dV^{+}/dI < 0 \qquad \text{and} \qquad dV^{-}/dI > R_0 L \tag{4.24}$$

These inequalities follow by proving the fact that $\pm(d/dz)(dV^{\pm}/dI) > 0$ for $z_{\min} \leqslant z \leqslant 1$ and by observing that dV^{+}/dI approaches zero while dV^{-}/dI approaches $R_0 L$ for $z \to 1$. It may be concluded from these inequalities that for Dirichlet boundary conditions the current is a single-valued function of the voltage for $V > 0$. The corresponding $I-V$ characteristic is plotted in Fig. 3.

We shall now prove that solutions of the form given in Eq. (4.2) are the only existing inhomogeneous solutions for Dirichlet boundary conditions. In view of the fact that the temperature is equal to zero at $y = 0$, the section adjacent to $y = 0$ is necessarily cold and has the temperature distribution given in Eq. (4.2a) for $0 \leqslant y \leqslant y_c$, where y_c is the first transition point along the wire. Similarly, the temperature for $y_c' \leqslant y \leqslant y_L$, where y_c' is the last transition point along the wire, has the form given in Eq. (4.2c). Since

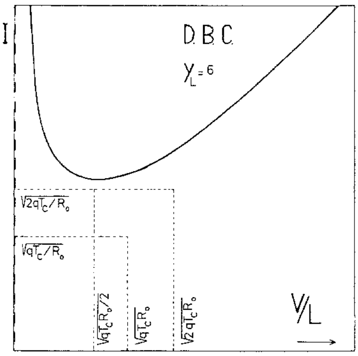

Fig. 3. The $I-V$ characteristic for Dirichlet boundary conditions for $y_L = 6$. The dashed lines correspond to the homogeneous solution and the solid line to the inhomogeneous solution.

$T(y_c) = T(y_c') = T_c$, it follows that $y_c' = y_L - y_c$ as in Eq. (4.2c). It remains to be shown that no transition points exist for $y_c < y < y_L - y_c$. This is equivalent to showing that no cold section can exist in this region. Assuming such a cold section to exist, the solution must be of the form

$$T(y) = (-2H)^{1/2} \cosh\left[y - \tfrac{1}{2}(y_c'' + y'''_c)\right] \tag{4.25}$$

where y_c'' and y'''_c are the transition points at the end of the cold section. Note that the amplitude is always determined by the constant value of H. Since H is necessarily positive according to Eqs. (4.2a) and (4.2c), and would have to be negative according to Eq. (4.25), it follows that no cold section can exist for $y_c < y < y_L - y_c$. Consequently, the wire is hot for $y_c < y < y_L - y_c$ and the solution must have the form given in Eq. (4.2b).

5. DISCUSSION: THE LONG WIRE

In the preceding sections we have derived analytic expressions for the nonuniform stationary-state temperature distributions as well as for the corresponding $I-V$ characteristics. This has been done both for Neumann and Dirichlet boundary conditions. This analysis has been performed for all values of the length of the wire. In practice, however, the wire is usually long in the sense that

$$L \gg (\lambda/q)^{1/2} \Leftrightarrow y_L \gg 1 \tag{5.1}$$

A typical value of y_L is 50.[2] It is therefore of interest to discuss the nature of the $I-V$ characteristics for this case in more detail.

We first consider inhomogeneous solutions for Neumann boundary conditions with periodicity $n = 1$. Using Eq. (3.21), one has in the limit of an infinite wire

$$\lim_{y_L\to\infty} \gamma(z; y_L) = \operatorname{sign} z \qquad \text{for} \quad z \neq 0 \tag{5.2}$$

Therefore one finds in the same limit [cf. Eq. (3.23)]

$$\lim_{y_L \to \infty} \left[V(I;L)/L \right] = \tfrac{1}{2} I R_0 (1 - \operatorname{sign} z) \qquad \text{for} \quad z \neq 0 \tag{5.3}$$

This implies that for $z > 0$, i.e., $I > (2qT_c/R_0)^{1/2}$, the $I-V$ characteristic becomes identical with the one for the uniform zero-temperature solution. Similarly, for $z < 0$, i.e., $(qT_c/R_0)^{1/2} < I < (2qT_c/R_0)^{1/2}$, it becomes identical with the one for the homogeneous $T(y) = T_h$ solution (cf. also Fig. 4). The asymptotic behavior is given by

$$\gamma(z; y_L) = \operatorname{sign} z \left(1 + y_L^{-1} \ln|z|\right) \qquad \text{for} \quad |z| e^{y_L} \gg 1 \tag{5.4}$$

and for the voltage

$$V(I;L) = \tfrac{1}{2} I R_0 L \left[1 - \operatorname{sign} z \left(1 + y_L^{-1} \ln|z|\right)\right] \qquad \text{for} \quad |z| e^{y_L} \gg 1 \tag{5.5}$$

In order to study the behavior of the $I-V$ characteristic for very small values of z it is convenient to consider the behavior of $z(\gamma; y_L)$ for large values of y_L. Inverting Eq. (3.21), one finds

$$\lim_{y_L \to \infty} z(\gamma; y_L) = \lim_{y_L \to \infty} (\sinh \gamma y_L / \sinh y_L) = 0 \qquad \text{for} \quad |\gamma| \neq 1 \tag{5.6}$$

Similarly, one finds [cf. Eq. (3.15)]

$$\lim_{y_L \to \infty} I(\gamma; y_L) = \lim_{y_L \to \infty} \left\{ 2qT_c/R_0 \left[1 - z(\gamma; y_L) \right] \right\}^{1/2}$$

$$= (2qT_c/R_0)^{1/2} \qquad \text{for} \quad |\gamma| \neq 1 \tag{5.7}$$

We note that keeping γ constant is equivalent to keeping V/IL constant. Equation (5.7) implies that the $I-V$ characteristic has a horizontal section where V/L varies between zero and $(2qT_cR_0)^{1/2}$, where I has the value given in Eq. (5.7) (cf. Fig. 4). This section of the $I-V$ characteristic corresponds essentially to the coexistence of two homogeneous phases, one cold ($T = 0$) and one hot ($T = T_h = 2T_c$), with variable relative lengths. The asymptotic behavior is given in this case by

$$z(\gamma; y_L) = e^{-(1-\gamma) y_L} - e^{-(1+\gamma) y_L} \qquad \text{for} \quad (1 - |\gamma|) y_L \gg 1 \tag{5.8}$$

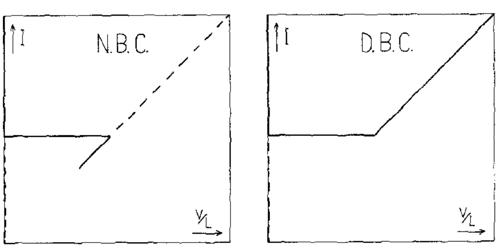

Fig. 4. The $I-V$ characteristics in the long-wire limit.

and for the current

$$I(\gamma; y_L) = \begin{array}{l} (2qT_c/R_0)^{1/2}\left(1 + \frac{1}{2}e^{-(1-\gamma)y_L} - \frac{1}{2}e^{-(1+\gamma)y_L}\right) \\ \text{for} \quad (1-|\gamma|)y_L \gg 1 \end{array} \tag{5.9}$$

For $n > 1$ the same analysis remains valid if one replaces y_L by y_L/n.

For Dirichlet boundary conditions we use Eq. (4.16) and find [see also Eq. (4.10)]

$$\lim_{y_L\to\infty} \gamma(z; y_L) = 1 \qquad \text{for} \quad z > \lim_{y_L\to\infty} z_{\min} = 0 \tag{5.10}$$

For the voltage [cf. Eq. (4.17)] one thus obtains

$$\lim_{y_L\to\infty}\left[V^{\pm}(I; L)/L\right] = \tfrac{1}{2}IR_0(1 \mp 1) \qquad \text{for} \quad z > 0 \tag{5.11}$$

The $I-V$ characteristic for V^+ therefore becomes identical with the one for the uniform zero-temperature solution, while the V^- characteristic becomes the one for the uniform $T(y) = T_h$ solution, both for $I > (2qT_c/R_0)^{1/2}$ [cf. Fig. 4]. The asymptotic behavior is given by

$$\gamma(z; y_L) = 1 + (2/y_L)\ln z \qquad \text{for} \quad ze^{y_L/2} \gg 1 \tag{5.12}$$

and for the voltage

$$V^{\pm}(I; L) = \tfrac{1}{2}IR_0\left\{1 \mp \left[1 + (2/y_L)\ln z\right]\right\} \qquad \text{for} \quad ze^{y_L/2} \gg 1 \tag{5.13}$$

For very small values of z we consider the following limit:

$$\lim_{y_L\to\infty} z(\gamma; y_L) = \lim_{y_L\to\infty}\left[\cosh(\tfrac{1}{2}\gamma y_L)/\cosh(\tfrac{1}{2}y_L)\right] = 0 \qquad \text{for} \quad \gamma \neq 1 \tag{5.14}$$

and similarly [cf. Eq. (5.7)]

$$\lim_{y_L\to\infty} I(\gamma; y_L) = (2qT_cR_0)^{1/2} \qquad \text{for} \quad \gamma \neq 1 \tag{5.15}$$

Therefore the $I-V$ characteristic has precisely the same horizontal section in this limit as in the Neumann case (cf. Fig. 4). The asymptotic behavior is now given by

$$z(\gamma; y_L) = e^{-(1-\gamma)y_L/2} + e^{-(1+\gamma)y_L/2} \qquad \text{for} \quad \tfrac{1}{2}(1-\gamma)y_L \gg 1 \tag{5.16}$$

and for the current

$$I(\gamma; y_L) = (2qT_c/R_0)^{1/2}\left(1 + \tfrac{1}{2}e^{-(1-\gamma)y_L/2} + \tfrac{1}{2}e^{-(1+\gamma)y_L/2}\right) \quad \text{for} \quad \tfrac{1}{2}(1-\gamma)y_L \gg 1 \tag{5.17}$$

It is interesting to note that the limit $y_L \to \infty$ may be compared to the so-called thermodynamic limit in equilibrium statistical mechanics. In equilibrium statistical mechanics it is necessary to take this limit in order to obtain the nonanalytic behavior of thermodynamic functions corresponding to equilibrium phase transitions. In the present case one must take the analogous limit $y_L \to \infty$ in order to obtain the nonanalytic behavior of V/L as a function of I corresponding to a "dynamical phase transition."

APPENDIX

In this appendix we consider the slightly more general model for which the resistivity is given by

$$R(T) = R_c + \tfrac{1}{2} R_0 \operatorname{sign}(T - T_c) \tag{A1}$$

where R_c and R_0 are positive constants such that $2R_c \geqslant R_0$. If $2R_c = R_0$ this model reduces to the one considered in the text. It is convenient to introduce the following parameter:

$$\begin{aligned} z(I) &\equiv (2R_c/R_0)(1 - qT_c/I^2R_c) \\ &= 1 - 2(T_c - T_f)/(T_h - T_f) \end{aligned} \tag{A2}$$

where

$$T_f \equiv q^{-1}I^2(R_c - \tfrac{1}{2}R_0), \qquad T_h \equiv q^{-1}I^2(R_c + \tfrac{1}{2}R_0) \tag{A3}$$

For the model in the text $T_f = 0$ and $T_h = q^{-1}I^2R_0$; consequently z reduces to the parameter introduced in Eq. (3.15). One may easily verify that the equation for the stationary state can be written as

$$\frac{d^2}{dy^2}\Delta T(y) - \Delta T(y) + T_c\left(\frac{2R_c}{R_0} - z\right)^{-1}\left[z + \operatorname{sign}\Delta T(y)\right] = 0 \tag{A4}$$

where

$$\Delta T(y) \equiv T(y) - T_c \tag{A5}$$

For $2R_c = R_0$ this equation is equivalent to Eq. (2.7) used for that case. One may now easily verify the following theorem:

Theorem. If $\Delta T(y)$ is a solution of Eq. (A4), then

$$\Delta T'(y) \equiv -\frac{(2R_c/R_0) - z}{(2R_c/R_0) + z}\Delta T(y) = -\left(\frac{I'}{I}\right)^2 \Delta T(y) \tag{A6}$$

is also a solution of Eq. (A4) with $z' = -z$.

This theorem is in general only useful for Neumann boundary conditions, which will also be satisfied by $T' = \Delta T' + T_c$ if they are satisfied by $T = \Delta T + T_c$. This is not the case for Dirichlet boundary conditions. In the transformation given in Eq. (A6) the lengths of the cold ($\Delta T < 0$) and the hot ($\Delta T > 0$) sections are interchanged. This is the "symmetry relation" given in Eq. (3.18).

Explicit solutions for the more general model, Eq. (A1), may easily be constructed by appropriately modifying the analysis in the text. In fact, Eq. (A4) may be rewritten in the form

$$\begin{aligned}\frac{d^2}{dy^2}\left[T(y) - T_f\right] - \left[T(y) - T_f\right] \\ + (T_h - T_f)\Theta((T(y) - T_f) - (T_c - T_f)) = 0\end{aligned} \tag{A7}$$

This equation is identical with Eq. (2.7) if one replaces T, T_h, and T_c by $(T - T_f)$, $(T_h - T_f)$, and $(T_c - T_f)$. The subsequent results may easily be modified accordingly.

In the general case also the cold section contributes to the voltage difference between the end points of the wire. This results in the existence of a second bifurcation point for Neumann boundary conditions for

$$I = \left[qT_c/(R_c - \tfrac{1}{2}R_0)\right]^{1/2}, \quad V = L\left[qT_c(R_c - \tfrac{1}{2}R_0)\right]^{1/2}$$

(note that $z = 1$ at this point). The other bifurcation point is given by

$$I = \left[qT_c/(R_c + \tfrac{1}{2}R_0)\right]^{1/2}, \quad V = L\left[qT_c(R_c + \tfrac{1}{2}R_0)\right]^{1/2}$$

($z = -1$ at this point). For the model treated in the text the first bifurcation point vanishes at infinity ($I \to \infty$ and $V \to 0$).

ACKNOWLEDGMENT

Dick Bedeaux, who suffered a terrible accident, wants to express his gratitude to neurosurgeon Dr. C. A. F. Tulleken for saving his ability to actively pursue his life as a scientist.

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